A Note on Convective Equilibrium

"*It's not even wrong*"

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I've always puzzled over the treatment of convection in climate modelings. The basic *Radiative Convective Equilibrium* (*RCM*) model supposes two parallel processes for transporting energy from the earth's surface. Should radiation be reduced by increased absorption, one might expect both processes to share in picking up the slack, their relative contributions a matter of theoretical interest.

The *a priori* assumption underlying climate models is the *Adiabatic Lapse Rate (ALR)*, a function wholly of a humidity-dependent heat capacity. Although rarely attributed, the hypothesis that this isentropic function might explain observed thermal gradients in the troposphere originated with none other than Lord Kelvin. At the January 1862 meeting of the Manchester Literary and Philosophical Society, a paper, *On the Convective Equilibrium of Temperature in the Atmosphere*, by Professor Wm. Thomson was read by Dr. Joule:

> *"When all the parts of a fluid are freely interchanged and not subject to the influence of radiation and conduction, the temperature of the fluid is said by the Author to be in a state of convective equilibrium."*

In May 1866, J.C. Maxwell responds: *On the Dynamical Theory of Gases,* Philosophical Transactions of the Royal Society of London, Vol. 157, p. 86 (1867),

> "T*he left-hand side of equation (147), as sent to the Royal Society, contained a term, the result of which was to indicate that a column of air, when itself, would assume a temperature varying with the height, and greater above than below. The mistake arose from an error in equation (143). Equation (147), as now corrected, shows that the flow of heat depends on the variation of temperature only, and not on the direction of the variation of pressure. A vertical column would therefore, when in thermal equilibrium, have the same temperature throughout."*

In October 1875, Ludwig Boltzmann writes: *Über das Wärmegleichgewicht von Gasen, auf welche äußere Kräfte wirken*, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen, Vol. 72-II, p. 443 (1876),

> *"Aus dieser Formel folgt, daß trotz der Wirksamkeit der äußeren Kräfte für die Richtung der Geschwindigkeit irgend eines der Moleküle jede Richtung im Raume gleich wahrscheinlich ist, ferner dass in jedem Raumelemente des Gases die schwindigkeitsvertheilung des Gases genau ebenso beschaffen ist, wie in einem Gase von gleicher Temperatur, auf das keine Aussenkräfte wirken. Der Effect der äusseren Kräfte besteht blos darin, dass sich die Dichte im Gase von Stelle zu Stelle verändert und zwar in einer Weise, welche schon aus der Hydrostatik bekannt ist."*

Presuming Messrs. Maxwell and Boltzmann yet retain some credibility regarding matters thermodynamic, how do we then interpret non-isothermal profiles? Obviously, as non-equilibria. Lacking external support, such states will relax towards an equilibrium and forestalling relaxation requires work, the dissipation of free energy, from external sources. Radiative fluxes dissipate by absorption, convective fluxes by viscosity.

RCE models offer a quite sophisticated calculation for radiative absorption but assume convection is implicitly incorporated with the *ALR*. Conventional justification is of a thermally-insulated parcel, *e.g.* a weather balloon, suspended in a fluid with an *ALR* profile. The parcel is in a state of neutral buoyancy and its static energy independent of altitude. But, translation at finite rates remains limited by viscous resistance. Our parcel has no preferred direction of motion and convective flux therefore nil.

The literature offers scant clues for analyzing dissipation in coupled, non-linear systems, so let us indulge in some DIY steady-state thermodynamics:

1. Replace extensive parameters in equilibrium thermodynamic formulae with their corresponding vector fluxes, $J_U = J_F + T J_S$. Temperature remains, as for equilibria, an integrating factor rendering fluxes state variables, independent of the paths by which steady states have evolved. Information as to a past history has been subsumed as entropy.
2. Dissipation is a global function defined as the rate at which free energy is consumed and given by the surface integral of J_F. To satisfy the 1st Law, the surface integral of J_U is zero.
3. The 2nd Law defines the local relationship of entropy and temperature, div $J_S = J_U$ dot grad(1/T).
4. Given prescribed surface temperatures, the internal thermal profile is then that which minimizes dissipation. For a 1-dimensional system, this is equivalent to the condition that internal deviations of J_U from its mean value be minimal.

Implementation requires positing local expressions relating radiative and conductive fluxes to temperature and altitude together with a variational function for temperature. As illustration, I've cobbled a short program mimicking a troposphere not entirely dissimilar to earth's. For polynomial T(z) functions with up to ten variable coefficients, the calculated profile appears nearly linear, although the actual lapse rate ranges 5.8 to 7.9 K/km. Numerical perturbations suggest a 0.79K surface increase for CO_2 doubling (3.7 W/m^2). With a constraint that changes in both boundary temperatures be identical, mimicking the *ALR*, 2.26K. Our thesis that thermal profiles ought be based on dissipation rather than equilibrium seems of some relevance to a prevalent orthodoxy. Anthropomorphically, nature seeks to accomplish tasks with minimal wasted energy, temperature being an essential tool in her kit.

https://pdq2021.000webhostapp.com/HBC_Model.pdf

Google's translation of Boltzmann's paragraph:

> *"From this formula, it follows that in spite of the effectiveness of the external forces for the direction of the velocity of any of the molecules, each direction in space is equally probable, furthermore that in each space element of the gas the velocity distribution of the gas is exactly the same as in a gas of same temperature, on which no external forces act. The effect of the external forces consists merely in the fact that the density in the gas changes from place to place in a manner which is already known from hydrostatics."*